NORTHERN ILLINOIS UNIVERSITY

Continued Fraction Sums and Products

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By

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Abstract

The focus of this project is to study the mathematical relationship of the *continued fraction sum* and the *continued fraction product*. Any real number can be written as a continued fraction, so the addition and multiplication of any two continued fractions is the same as the real numbers they represent. However, the *continued fraction sum* and *continued fraction product* each result in another continued fraction. The problem is the determining the relationship that this new continued fraction has with the original two. This will be achieved by observing the characteristics of the 3D graphs contained within the unit cube. This topic is interesting because it is relatively unexplored. Whether or not these relationships will lead to anything significant is yet to be determined.

Section 1
An Introduction to Continued Fractions

We begin by defining the general form of a continued fraction. It is the expression of the form

$$a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\cfrac{\ddots}{+\cfrac{b_{n-2}}{a_{n-1}+\cfrac{b_{n-1}}{a_n+\dots}}}}}$$

where each a_j and b_j is a real or complex number.

Definition 1: An continued fraction is called "simple" if all of the following are true:

(i) $b_i = 1$ for $i \geq 1$
(ii) a_i is a positive integer for $i \geq 2$
(iii) a_1 is an integer.

The expression for the general form of a simple continued fraction is given by

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{\ddots}{+\cfrac{1}{a_{n-1}+\cfrac{1}{a_n+\dots}}}}}$$

Definition 2: Each a_i of a continued fraction is called the *partial quotient*.

Definition 3: If a continued fraction is simple and has finitely many partial quotients, then it is called a *finite simple continued fraction*.

The expression for the general form a finite simple continued fraction is given by

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots+\cfrac{1}{a_{n-1}+\cfrac{1}{a_n}}}}$$

The finite simple continued fraction is be denoted $[a_0;a_1,\dots,a_n]$ and the infinite simple continued fraction is be denoted $[a_0;a_1,\dots]$.
Note: The continued fractions used in the following sections will be containded in the unit cube, where $a_0=0$. The a_0 term will be dropped and the continued fractions written as $[a_1,\dots,a_n]$.

Theorem 4: Any rational number can be expressed as a finite simple continued fraction.

Proof: Let $\frac{a}{b}$ be any rational number where b > 0. Using Euclid's algorithm, we get

$$\begin{array}{lll} a=b*a_1+r_1 & \text{for} & 0<r_1<b \\ b=r_1*a_2+r_2 & \text{for} & 0<r_2<r_1 \\ r_1=r_2*a_3+r_3 & \text{for} & 0<r_3<r_2 \\ \vdots & & \\ r_{n-3}=r_{n-2}*a_{n-1}+r_{n-1} & \text{for} & 0<r_{n-1}<r_{n-2} \\ r_{n-2}=r_{n-1}*a_n & & \end{array}$$

Thus

$$\frac{a}{b}=a_1+\frac{r_1}{b}=a_1+\frac{1}{b/r_1}$$

$$\frac{b}{r_1}=a_2+\frac{r_2}{r_1}=a_2+\frac{1}{r_1/r_2}$$

$$\frac{r_1}{r_2}=a_3+\frac{r_3}{r_2}=a_3+\frac{1}{r_2/r_3}$$

$$\vdots$$

$$\frac{r_{n-3}}{r_{n-2}}=a_{n-1}+\frac{r_{n-1}}{r_{n-2}}=a_{n-1}+\frac{1}{r_{n-2}/r_{n-1}}$$

$$\frac{r_{n-2}}{r_{n-1}}=a_n\ .$$

Using successive substitution, it follows that

$$\frac{a}{b}=[a_1,a_2,\dots,a_n]\ .$$

Section 2
Addition of Continued Fractions

Definition 5: Let $\alpha=[a_1, a_2, ...]$ and $\beta=[b_1, b_2, ...]$ be simple infinite continued fractions. Then the *continued fraction sum* of these two continued fractions is defined to be

$$\alpha \oplus \beta=[a_1+b_1, a_2+b_2, ...] \ .$$

Definition 6: Let $\alpha=[a_1, a_2, ..., a_n]$ and $\beta=[b_1, b_2, ..., b_m]$ be simple finite continued fractions.

If m = n, then the *continued fraction sum* of these two continued fractions is defined as

$$\alpha \oplus \beta=[a_1+b_1, a_2+b_2, ..., a_n+b_n] \ .$$

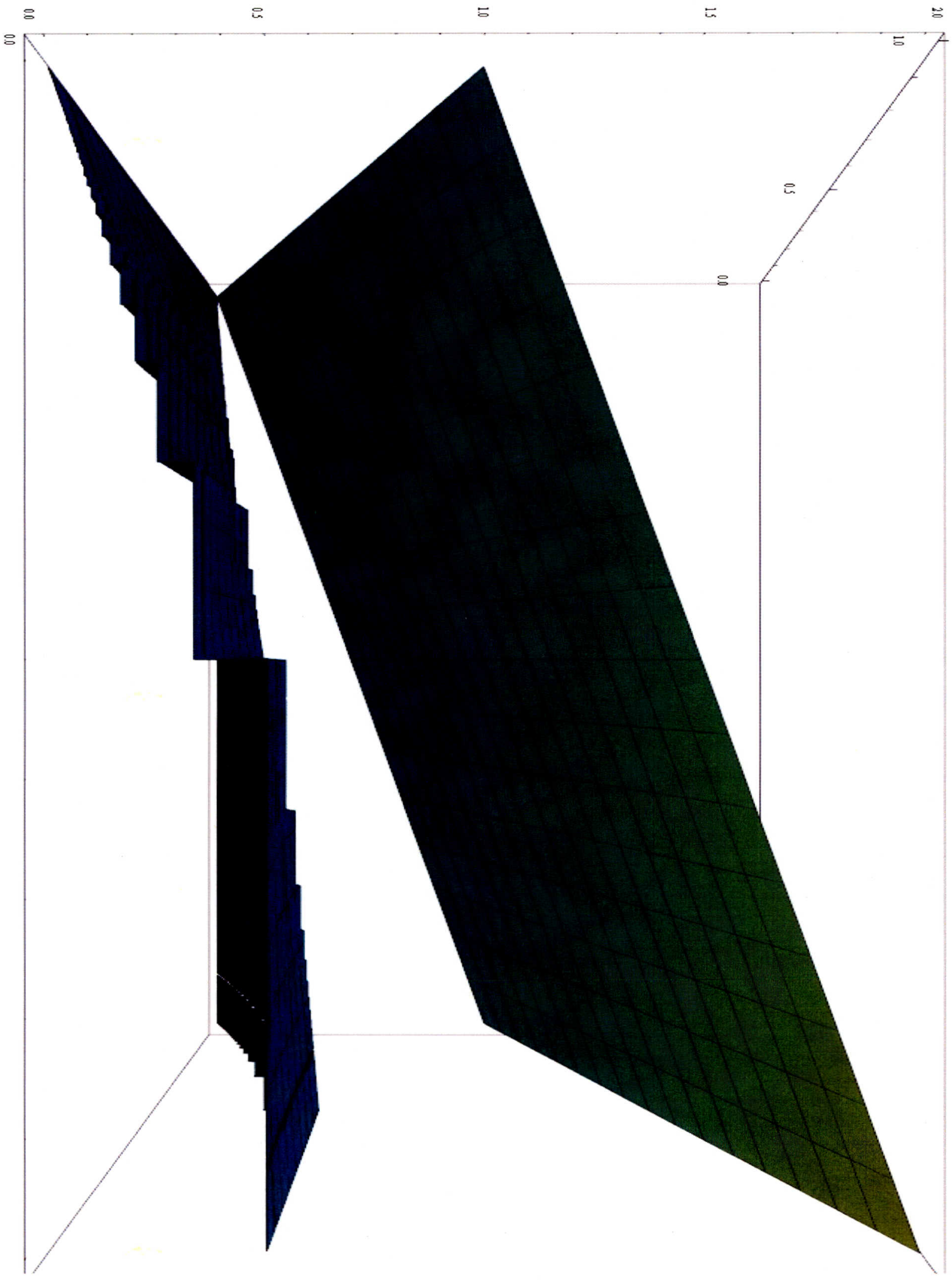
If m > n, then the *continued fraction sum* is $\alpha \oplus \beta=[a_1+b_1, a_2+b_2, ..., a_n+b_n, b_{n+1}, ..., b_m]$.

Now the sum of two continued fractions is not equal to the sums of the numbers they represent. So the question remains, is there a relationship between the continued fraction sum and the addition of the numbers they represent?

We will contemplate this question first by getting a graphical representation of the continued fraction product. This will be achieved by using the program Mathematica.

The graph will be contained within the unit cube. Since writing zero as a continued fraction poses problems we will make the starting point a random number very close to zero. Mathematica will not perform continued fraction addition when the number of partial quotients differ. To compensate for this , the random number will have a large number of decimal digits to ensure at least 20 partial quotients in the continued fraction. The intervals between plot points will be $\frac{1}{211}$, since 211 is prime, which will help with the randomization of plot points.

The following pictures are all different graphs, specifically different initial points, which are viewed from different perspectives. The flat plane is the graph of $x+y$ for the numbers that the continued fractions represent.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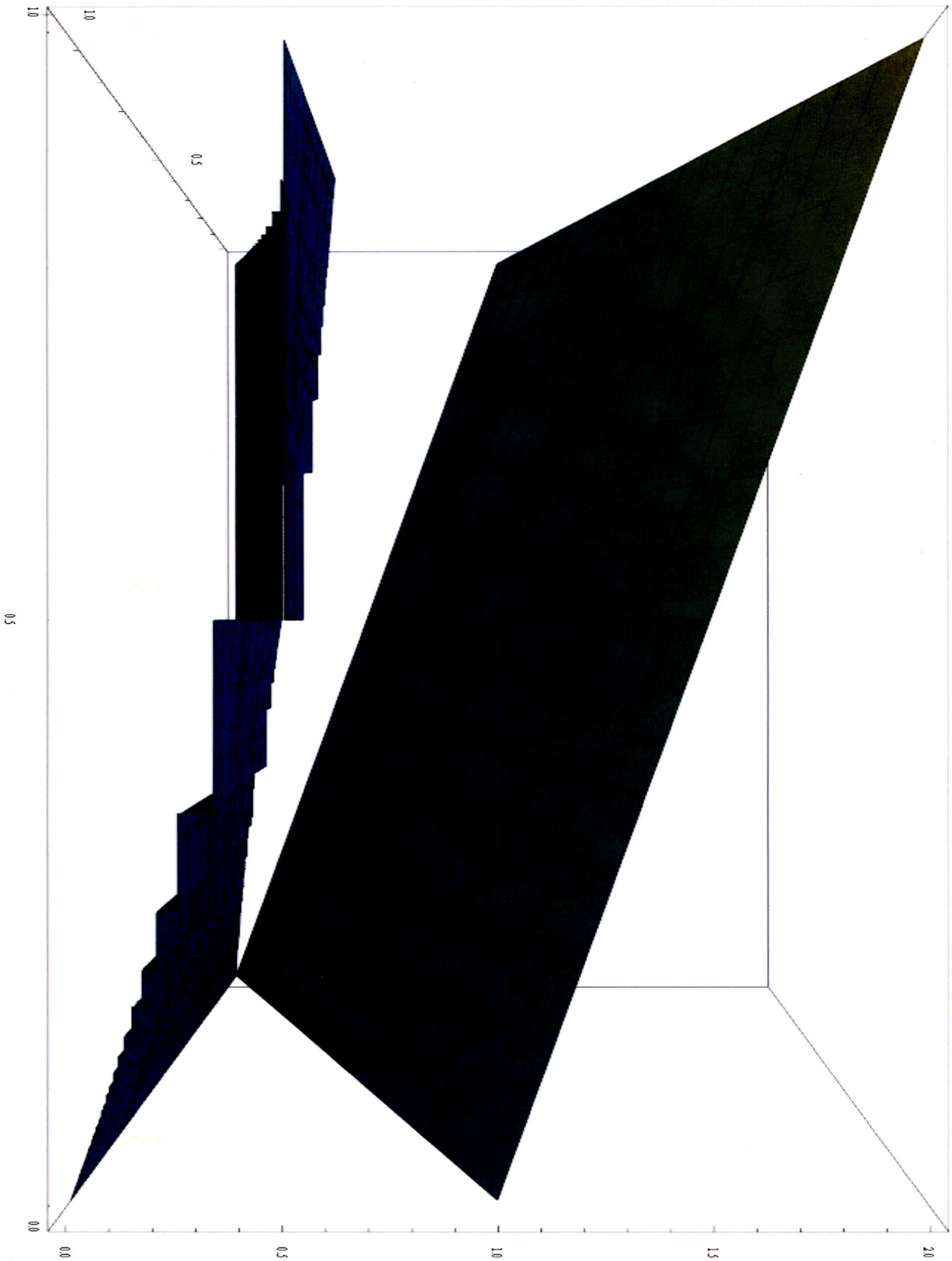
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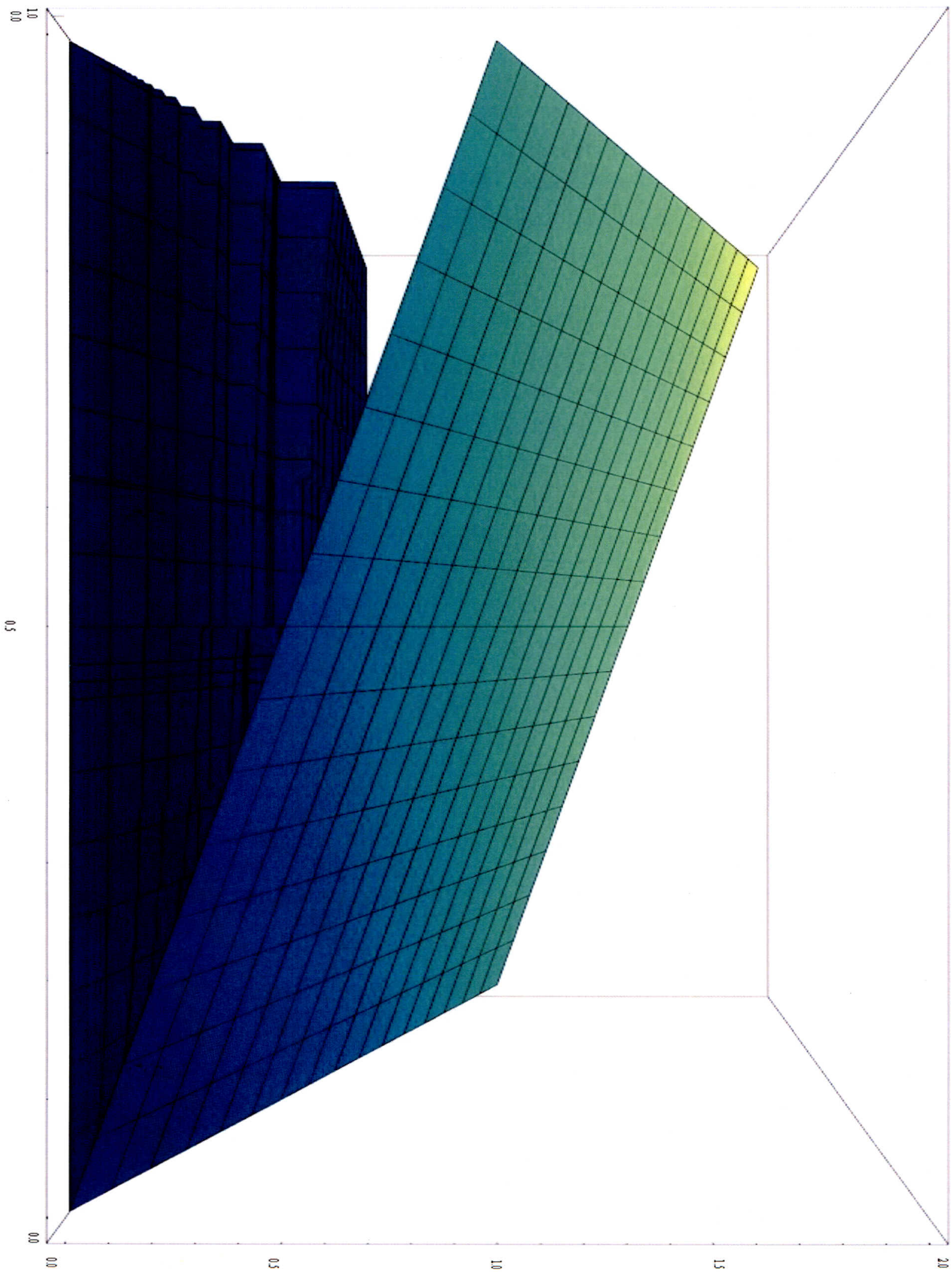
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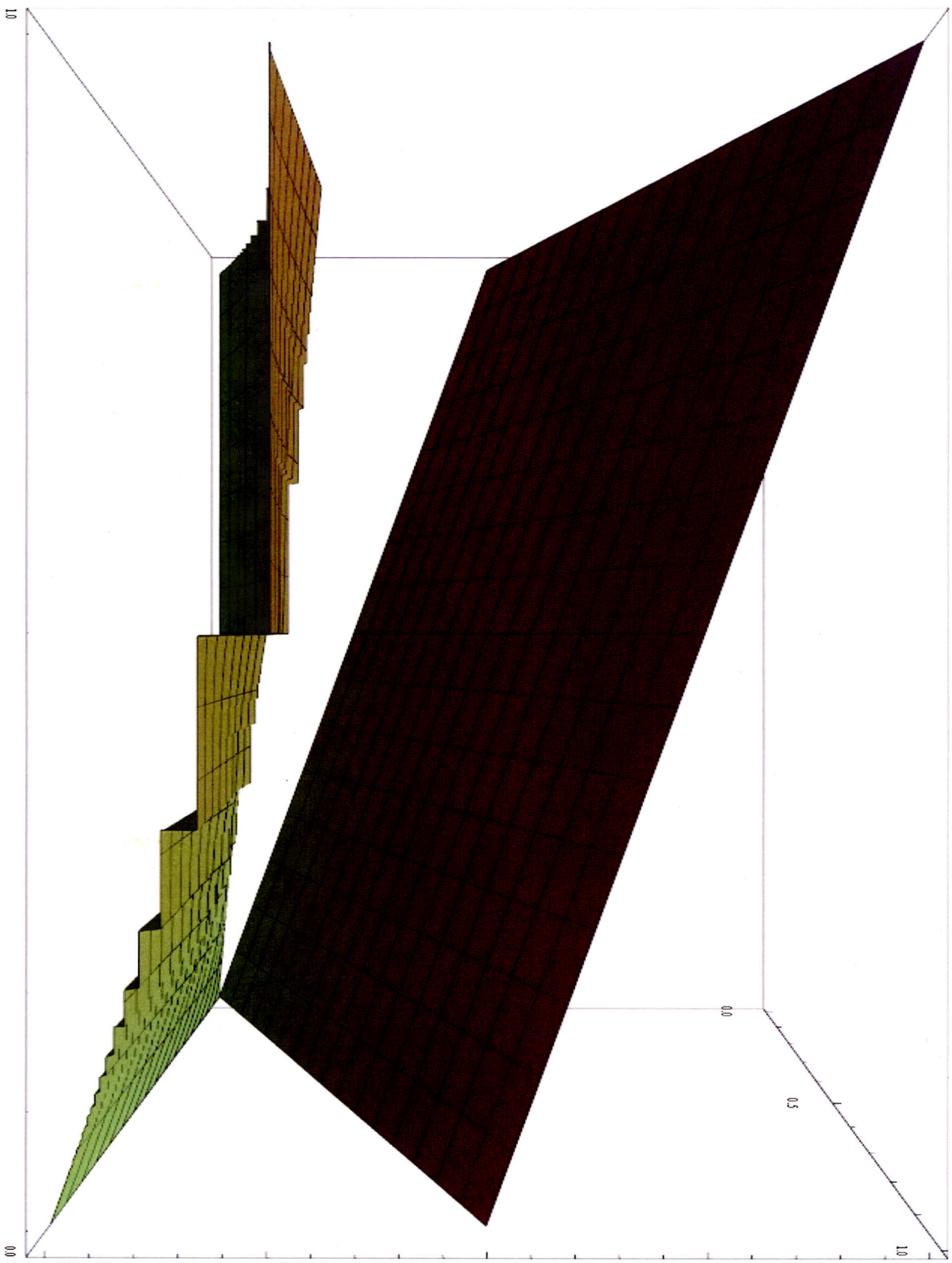
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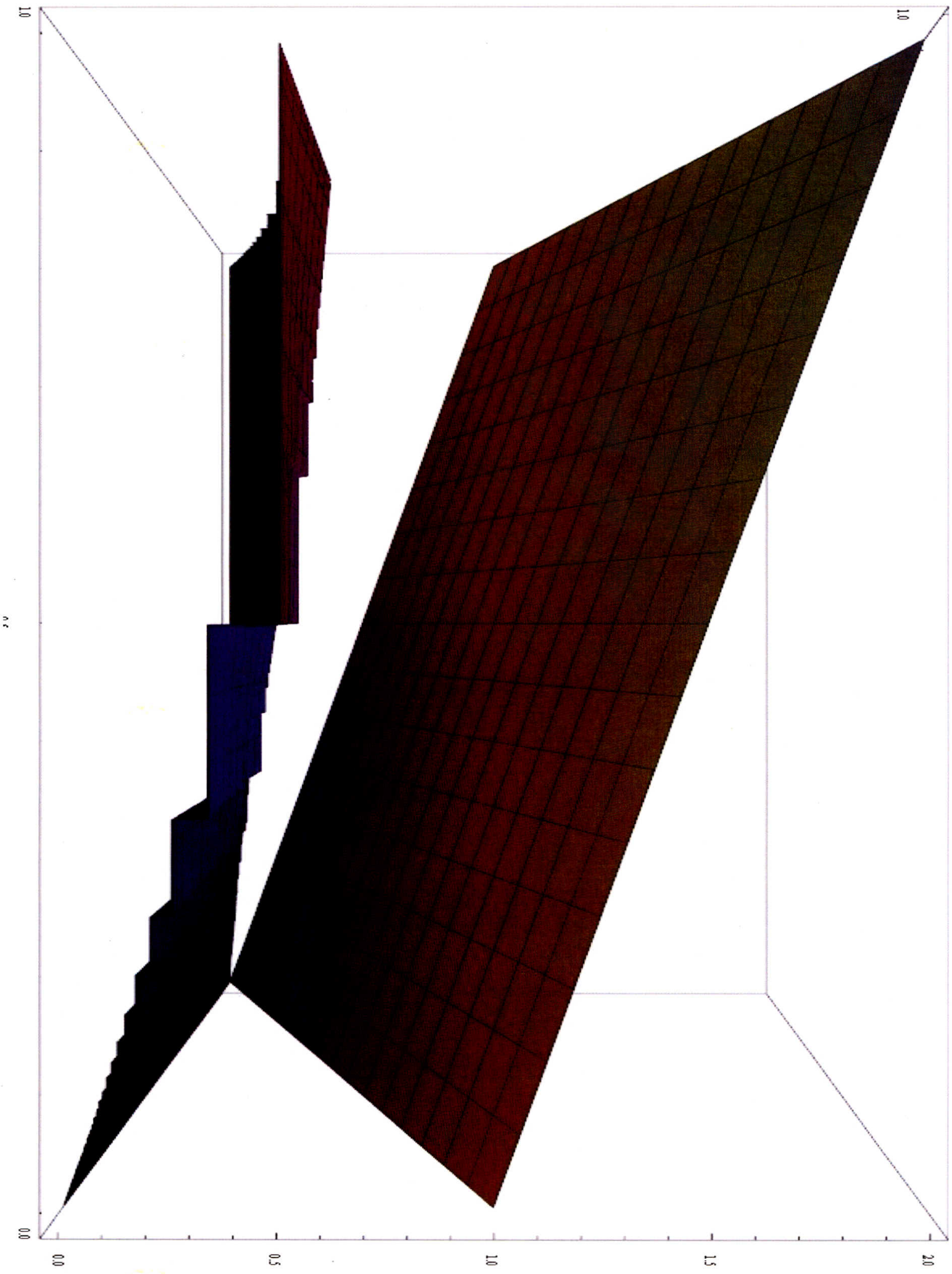
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These graphs are symmetric fractal patterns, and they can be explained by the partial quotients of the continued fraction. When looking along the x or y axis we observe that there are discontinuities, ('jumps'), in the graph of $x \oplus y$. Going along either axis from 1 to 0 we see that these jumps appear to occur at $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... etc.

Let $\alpha=[0, a_2, \ldots, a_{20}]$, $\beta=[0, b_2, \ldots, b_{20}]$ be representative of the plot points in the graph, where α's are the point on the x-axis and β's are the points along the y-axis, let $\gamma=\alpha \oplus \beta$. The partial quotients a_1 and b_1 are zero because the plot points are contained within the unit square and therefore never exceed a value of 1.

Now, consider only the first, nonzero partial quotient of each continued fraction. So $\alpha=[0, a_2]$, $\beta=[0, b_2]$ and the first partial approximates are $\alpha=\frac{1}{a_2}$ and $\beta=\frac{1}{b_2}$. Then the continued fraction sum is $\frac{1}{a_2+b_2}$. Now we know that as a_2+b_2 increases that $\frac{1}{a_2+b_2}$ decreases. So, if we fix $b_2=1$ and increment a_2 we see that $\frac{1}{a_2+b_2}$ = { $\frac{1}{2}$, $\frac{1}{3}$, ... , $\frac{1}{n+1}$ } for a_2 = {1,2, ... , n}. In general, $c_2=k$ for $k \leq \frac{1}{\gamma} < k+1$; i.e., $\frac{1}{k+1} < \gamma \leq \frac{1}{k}$, and the first approximate is $\gamma \approx \frac{1}{a_2+b_2}$.

Thus the most prominent behavior of the graph is explained, but what of the more subtle behavior? Recall that the continued fraction plot points used 20 partial quotients. Now increase the number of partial quotients that are considered.

So, fix $c_2=k$ where k is some constant, so we can describe the behavior of the graph between discontinuities of the first approximate. As a result, all the continued fractions in this interval can be written as $\gamma=\cfrac{1}{k+\cfrac{1}{r_3}}$, where $1 \leq r_3 < \infty$ and $c_2=[r_3]$. As r_2 increases from 1 to ∞ , γ increases from $\frac{1}{k+1}$ to $\frac{1}{k}$, thus taking all values in the given interval. So, in general, $c_2=l$, for $l \leq r_2 < l+1$, i.e.,

$$\frac{1}{k+\frac{1}{l}} \leq \gamma < \frac{1}{k+\frac{1}{l+1}} .$$

From this we can see that the second approximate of the sum of two continued fractions is

$$\gamma \approx \cfrac{1}{a_1+b_1+\cfrac{1}{a_2+b_2}} .$$

The results for the successive approximations are similar, and we see that for each approximation, the behavior on the fixed interval is the same as the behavior of the whole graph.

Section 3
Multiplication of Continued Fractions

Definition 7: Let $\alpha=[a_1, a_2, ...]$ and $\beta=[b_1, b_2, ...]$ be simple infinite continued fractions. Then the *continued fraction product* of these two continued fractions is defined to be

$$\alpha \otimes \beta=[a_1 * b_1, a_2 * b_2, ...] \ .$$

Definition 8: Let $\alpha=[a_1, a_2, ..., a_n]$ and $\beta=[b_1, b_2, ..., b_m]$ be simple finite continued fractions with $m \geq n$.

If m = n, then the *continued fraction product* of these two continued fractions is defined to be

$$\alpha \otimes \beta=[a_1 * b_1, a_2 * b_2, ..., a_n * b_n] \ .$$

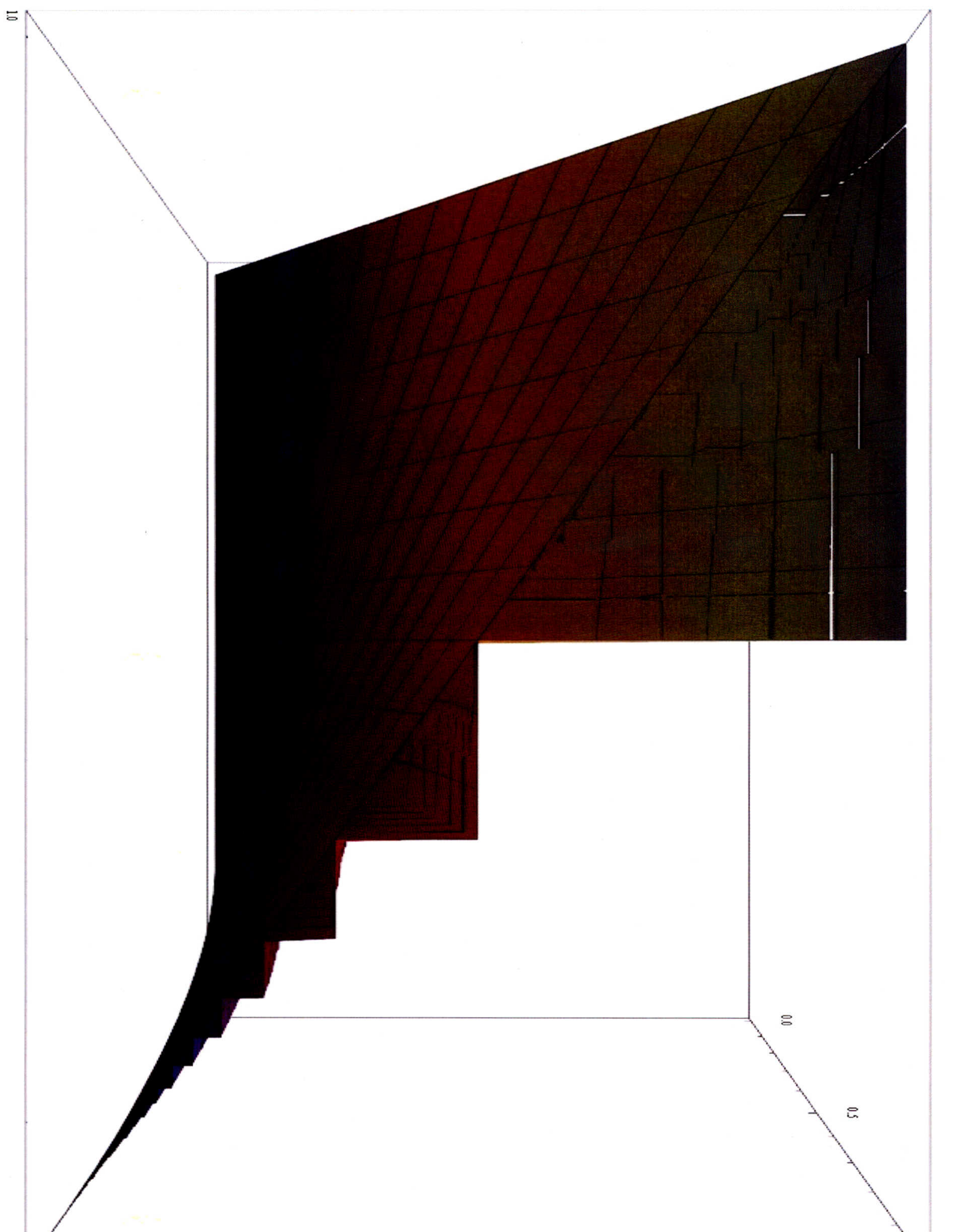
If m > n, then the *continued fraction product* is $\alpha \otimes \beta=[a_1 * b_1, a_2 + b_2, ..., a_n + b_n]$.

Like the sum, the product of two continued fractions is not equivalent to the products of the numbers they represent. So is there a relationship?

Once again, we'll start by looking at graphical representations of the continued fraction products.

The parameters for these graphs will be the same as those used for the sum, but will be repeated again for ease of reference. These graphs will be contained within the unit cube. Since writing zero as a continued fraction poses problems we will make the starting point a random number very close to zero. Mathematica will not perform continued fraction addition when the number of partial quotients differ. To compensate for this , the random number will have a large number of decimal digits to ensure at least 20 partial quotients in the continued fraction. The intervals between plot points will be $\frac{1}{211}$, since 211 is prime, which will help with the randomization of plot points.

The following pictures are all different graphs, specifically different initial points, which are viewed from different perspectives. The flat plane is the graph of $x * y$ for the numbers that the continued fractions represent.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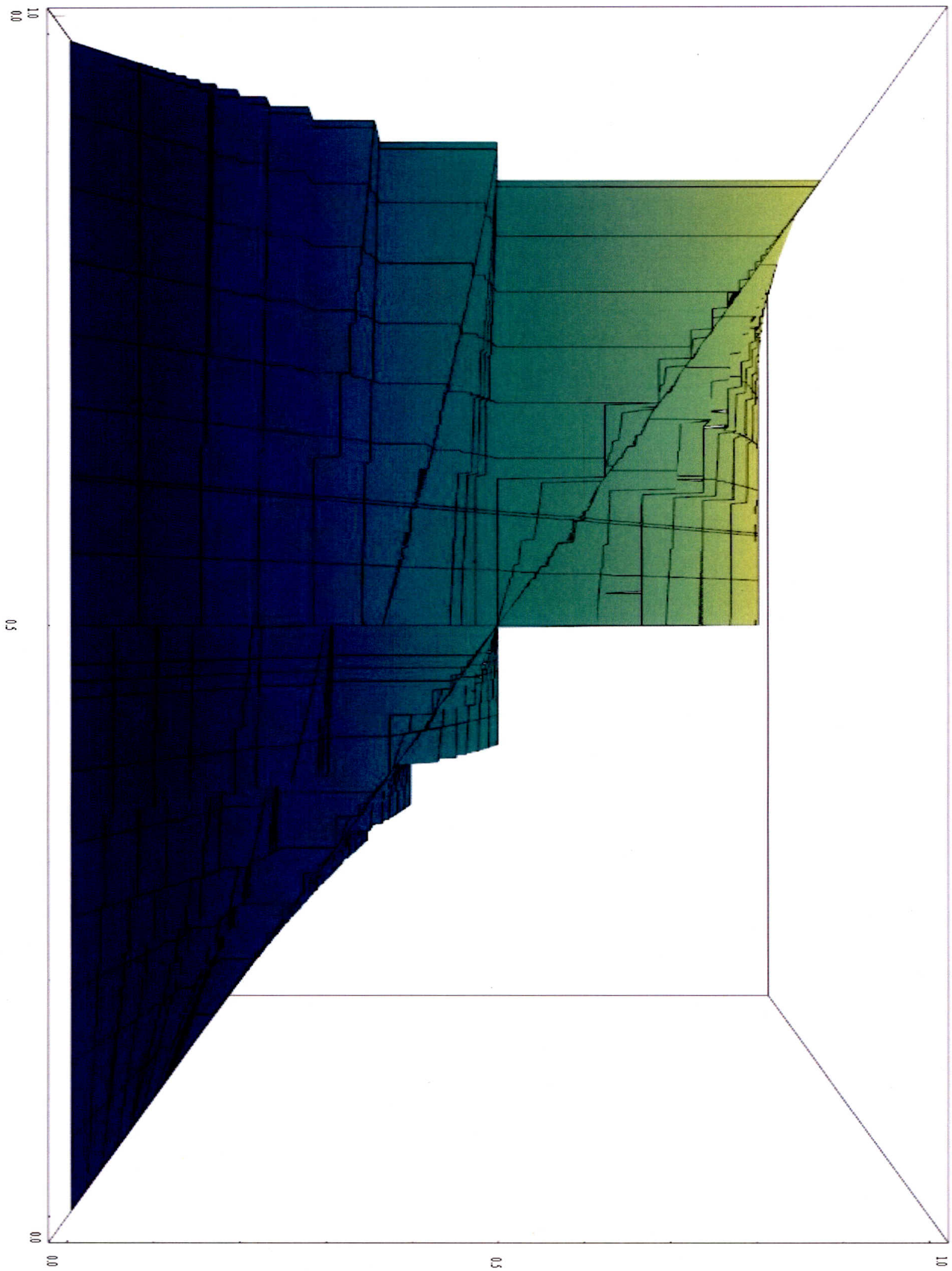
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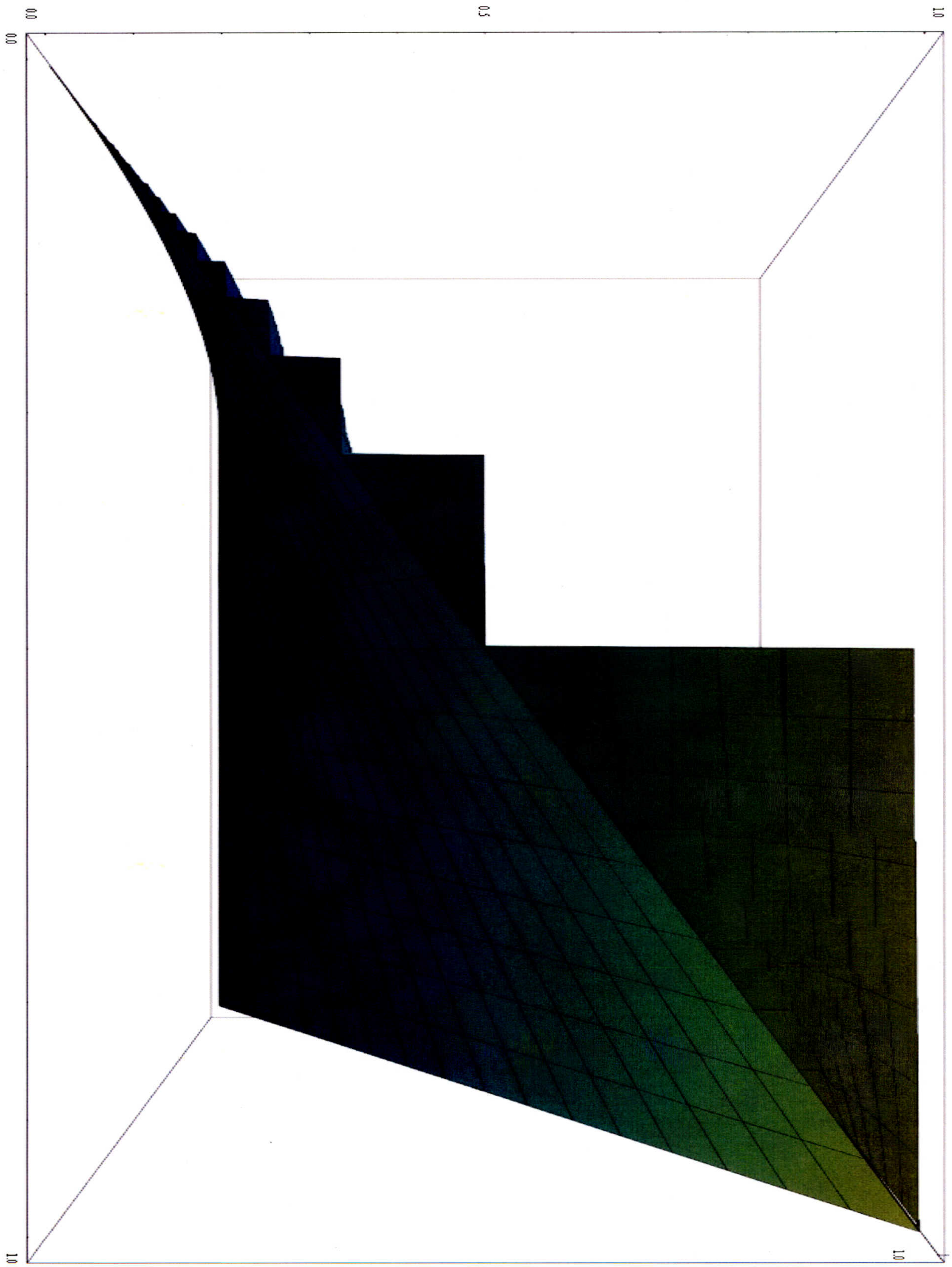
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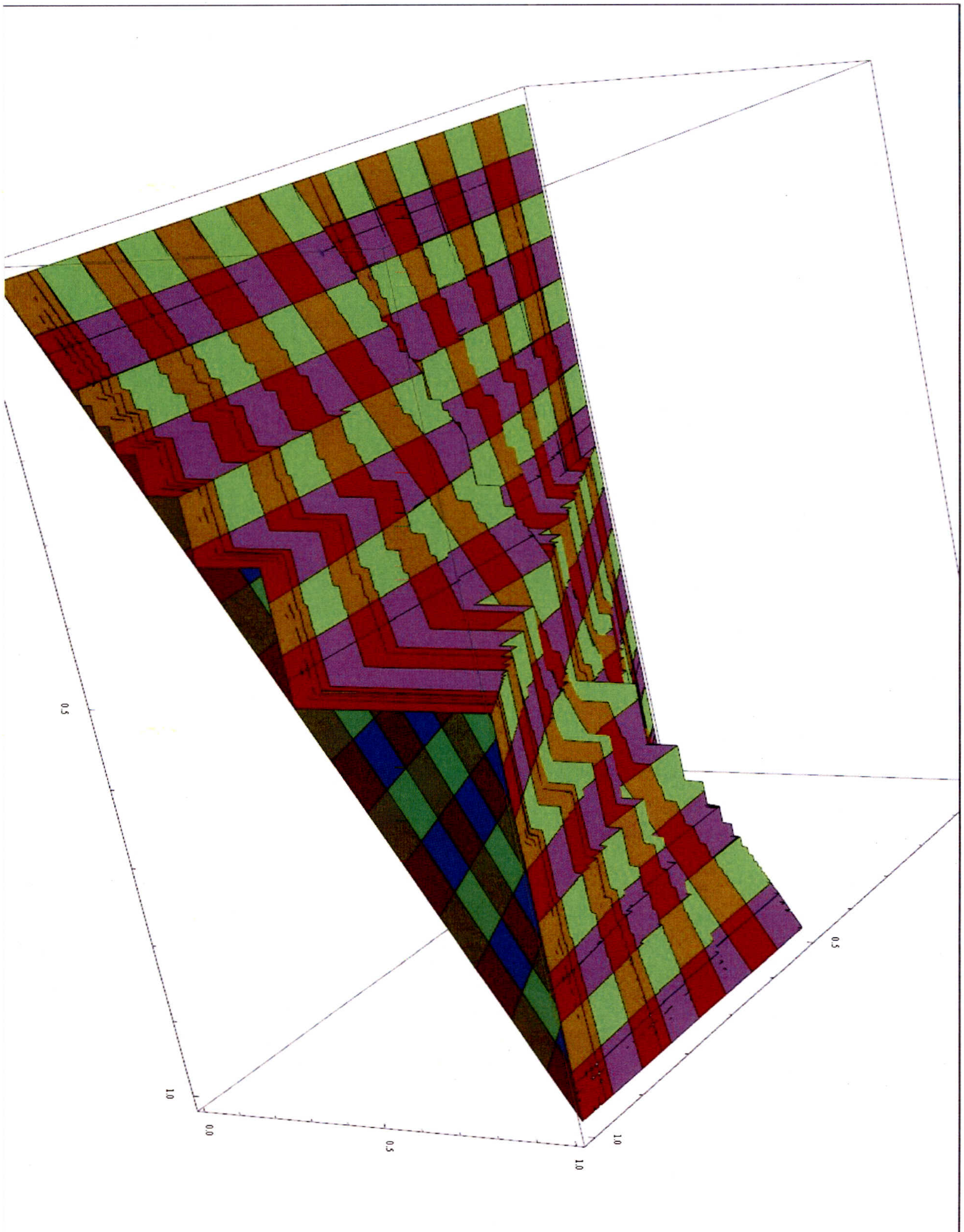
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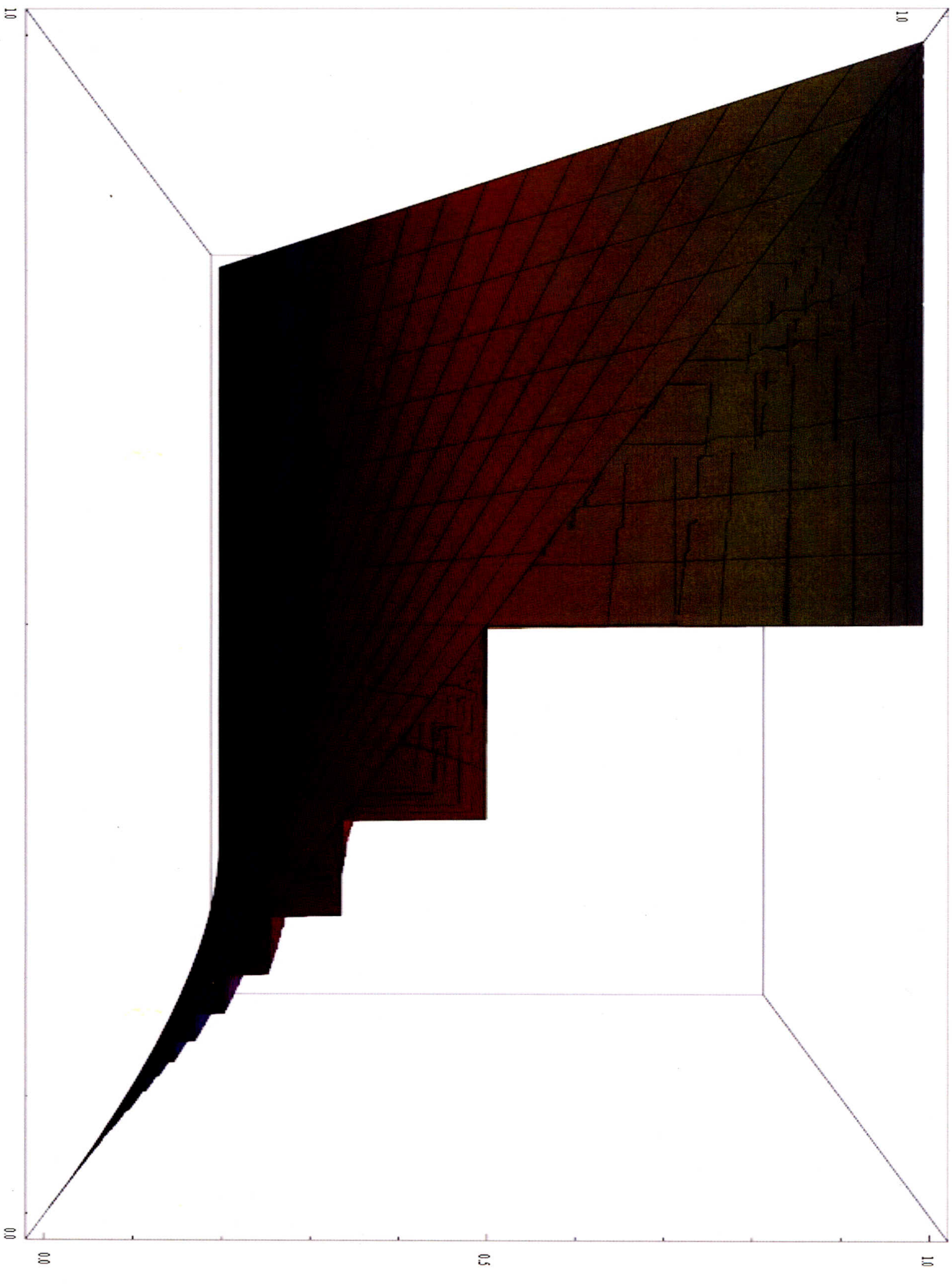
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As with addition, these graphs are symmetric fractal patterns, and they can be explained by the partial quotients of the continued fraction, and once again when looking along the x or y axis we observe the presence of discontinuities in the graph of $x \otimes y$. These jumps still appear to occur at $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, … etc.

So, let $\alpha=[0,a_2,\dots,a_{20}]$ and $\beta=[0,b_2,\dots,b_{20}]$ be representative of the plot points in the graph, where α's are the point on the x-axis and β's are the points along the y-axis. Again, the partial quotients a_1 and b_1 are zero because the plot points are contained within the unit square and therefore never exceed a value of 1.

Again, consider only the first, nonzero partial quotient of each continued fraction. So $\alpha=[0,a_2]$, $\beta=[0,b_2]$ and the first partial approximates are $\alpha=\frac{1}{a_2}$ and $\beta=\frac{1}{b_2}$. Then the continued fraction sum is $\frac{1}{a_2 * b_2}$, and as $a_2 * b_2$ increases then $\frac{1}{a_2 * b_2}$ decreases. So, fix $b_2=1$ and increment a_2 we see that $\frac{1}{a_2 * b_2}$ = { 1 , $\frac{1}{2}$, $\frac{1}{3}$, … , $\frac{1}{n}$ } for a_2 = {1,2,3, … , n}. This explains the most prominent behavior of the graph.

The further behavior is explained in the same way as the material in chapter 2. However, the resulting approximations are $\gamma \approx \frac{1}{a_1 * b_1}$ for the first approximation and

$$\gamma \approx \frac{1}{a_1 * b_1 + \frac{1}{a_2 * b_2}}$$

for the second, etc.

Section 4
Multiplication Families of Continued Fractions

An interesting concept to consider is the behavior of the multiplication of continued fractions when one of them is fixed.

Let α and β both be infinite simple continued fractions. Now set $\alpha=\sqrt{2}=[1,\bar{2}]$ and $\beta=\sqrt{n}$ such that β is the continued fraction of $\sqrt{n}$.

Then as we increment n, we get the following:

$$\sqrt{2}*\sqrt{3}=[1,\overline{2,4}]=\sqrt{6}-1$$

$$\sqrt{2}*\sqrt{4}=[2,\bar{0}]=2$$

$$\sqrt{2}*\sqrt{5}=[2,\bar{8}]=\sqrt{17}-2$$

$$\sqrt{2}*\sqrt{6}=[2,\overline{4,8}]=\sqrt{18}-2$$

$$\sqrt{2}*\sqrt{7}=[2,\overline{2,2,2,8}]=\frac{1}{2}\sqrt{78}-2$$

$$\sqrt{2}*\sqrt{8}=[2,\overline{2,4}]=2\sqrt{5}-2$$

$$\sqrt{2}*\sqrt{9}=[3,\bar{0}]=3$$

$$\sqrt{2}*\sqrt{10}=[3,\overline{1,2}]=\sqrt{37}-3$$

$$\sqrt{2}*\sqrt{11}=[3,\overline{6,12}]=\sqrt{38}-3$$

$$\sqrt{2}*\sqrt{12}=[3,\overline{4,12}]=\sqrt{39}-9$$

$$\sqrt{2}*\sqrt{13}=[3,\overline{2,2,2,2,1,2}]=\frac{1}{29}\sqrt{34597}-\frac{87}{29}$$

$$\sqrt{2}*\sqrt{14}=[3,\overline{2,4,2,1,2}]=\frac{4}{5}\sqrt{65}-3$$

$$\sqrt{2}*\sqrt{15}=[3,\overline{2,1,2}]=\sqrt{42}-3$$

So, we see a pattern that arises, specifically for β that can be written as $\beta=\sqrt{m^2-1}$. With these β the results of $\alpha*\beta$, for $\alpha=\sqrt{2}$, are given by $\sqrt{2}*\sqrt{m^2-1}=\sqrt{4*b_1^2+m^2-1}-n-b_1$ where n is the greatest perfect square less than $\sqrt{m^2-1}$ and b_1 is the first partial quotient of β. Other families for α of the form $\alpha=\sqrt{n}$ yield similar results.

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